J Stat Phys (2009) 136: 103–115
DOI 10.1007/s10955-009-9764-7

Regulation Mechanisms in Spatial Stochastic Development Models

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Received: 4 September 2008 / Accepted: 26 May 2009 / Published online: 12 June 2009

Abstract The aim of this paper is to analyze different regulation mechanisms in spatial continuous stochastic development models. We describe the density behavior for models with global mortality and local establishment rates. We prove that the local self-regulation via a competition mechanism (density dependent mortality) may suppress a unbounded growth of the averaged density if the competition kernel is superstable.

Keywords Continuous systems · Spatial birth-and-death processes · Correlation functions · Establishment · Density dependent mortality · Development models

1 Introduction

We will discuss some classes on interacting particle systems (IPS) located in the Euclidean space $\mathbb{R}^d$. The phase space of such system is the configuration space $\Gamma = \Gamma(\mathbb{R}^d)$ on $\mathbb{R}^d$. By definition, each configuration $\gamma \in \Gamma$ is a locally finite subset $\gamma \subset \mathbb{R}^d$. So, due to the standard terminology, we will deal with continuous systems. Random evolutions of IPS are given by Markov processes on Γ. Between all such processes, one may distinguish a subclass of so-called spatial birth-and-death Markov processes. In these processes points of a randomly evolving configuration appear and disappear due to a Markov rule (see (4) below). Particular types of spatial birth-and-death processes are motivated by several applications. For example, Glauber type dynamics for classical continuous gases belongs to this class [1, 11]. Another very essential source of such processes is given by individual based models in spatial ecology or agent based models in socio-economic systems, see, e.g., [3] and the references therein. In any case, a concrete form of birth and death rates in the stochastic dynamics should reflect a microscopic structure of the system under consideration.

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To describe the problem we are going to analyze, let us start with the simplest case of a pure birth stochastic Markov process. In this process, new points appear in the configuration independently of existing points and locations of these new points are uniformly distributed in the space. A possible interpretation of such random evolution is related to an independent creation of identical economic units in the space without any influence of their spatial locations. We will call this process the free stochastic development model. Another motivation to study such stochastic evolutions comes from applications of the free development dynamics to generalized mutation-selection models in mathematical genetics. In these models a configuration describes locations of mutations inside of a genom and new mutations spontaneously appear and are equally distributed in the genom, see [7, 13, 19]. Obviously, this process is monotonically growing and it is easy to show that the density of particles in such a system will linearly grows in time. We would like to answer the following question: How may global regulations and local interactions change the asymptotic behavior of the system? More precisely, we will consider three particular cases of stochastic development models including:

(i) A global regulation via a mortality rate that prescribes to particles (economic units) random life times (exponentially i.i.d. with a parameter $m > 0$). This case corresponds to the well-known Surgailis independent birth-and-death Markov processes on Γ, see, e.g., [12, 20, 21]. In the framework of mathematical physics, it is just Glauber dynamics for classical free gas.

(ii) An establishment effect. In this case, the distribution of the position of a new particle depends on the local structure of the configuration. Newborn units will appear with small intensity in densely occupied regions. We will see that the establishment itself is not enough to prevent the growth of the density in the system. In fact, the establishment effects slower (logarithmic) growth contrary to the linear growth in the free model.

(iii) A self-regulation via competition. The competition is reflected in a density dependent mortality. The latter means that the mortality rate for each unit depends on the local structure of the configuration around this unit. The mortality rate enters into the model as a relative energy of a particle inside the configuration corresponding to a competition potential. The described competition mechanism provides only a local regulation of dense regions inside a configuration. Nevertheless, Theorem 1 shows a global bound for the averaged density of the stochastic development process. Note that for the proof of this result we use an assumption of positive definiteness (and, as a consequence, superstability) of the competition potential in the form stated in [16]. Therefore, the main result concerning the competition case may be read as follows: a properly organized competition in the stochastic development systems produces a self-regulation for the density of units.

Let us stress an essential point concerning the main aim of this paper. At present, we have quite restrictive conditions for the existence of general spatial birth-and-death processes, see e.g. [5]. In many applications we need a weaker information. Namely, we are interesting in the existence of Markov functions corresponding to given birth and death rates and a certain class of initial distributions on Γ. These Markov functions and their one-dimensional distributions are enough to describe the time evolution of initial states of systems and to analyze asymptotic properties of stochastic dynamics (invariant states, ergodicity etc.). In the case of infinite particle systems, the concept of Markov functions is strictly weaker than that of Markov processes, and for particular models considered below there exist constructive methods which solve the existence problem, see [3, 5, 10] and [9]. But the main aim of our analysis is to obtain an *a priori* information about the time-space behavior of such important characteristics of these processes as correlations functions of their one-dimensional distributions which are probability measures on Γ. In particular, we are interested in the behavior of the particle density in course of the stochastic evolution. A constructive possibility to obtain some *a priori* bounds on characteristics of Markov dynamics may be also realized in other interesting IPS. As an example, we can mention the Dieckmann–Law model in spatial ecology where the existence problem remains open but conditions for explosion and non-explosion are stated in terms of the parameters of the model in [2]. Moreover, analogously to the well-known situation in the PDE theory, *a priori* bounds may play a crucial role in the study of the existence problem.

2 General Facts and Notations

Let $\mathcal{B}(\mathbb{R}^d)$ be the family of all Borel sets in $\mathbb{R}^d$ and $\mathcal{B}_b(\mathbb{R}^d)$ denotes the system of all bounded sets from $\mathcal{B}(\mathbb{R}^d)$.

We denote $\Gamma^{(n)}$ the space of all n-point subsets (configurations) from $\mathbb{R}^d$, $n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$. As a set $\Gamma^{(n)}$ is equivalent to the symmetrization of $(\mathbb{R}^d)^n$ without diagonals, $\Gamma^{(0)} := \{\emptyset\}$. Then one can introduce the corresponding topology and Borel σ-algebra. Also one can define a measure $m^{(n)}$ as an image of the product of Lebesgue measures $dm(x) = dx$ on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$.

The space of finite configurations $\Gamma_0 := \bigsqcup_{n\in\mathbb{N}_0} \Gamma^{(n)}$ is equipped with the topology which has structure of disjoint union. Therefore, one can define the corresponding Borel σ-algebra $\mathcal{B}(\Gamma_0)$. For $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ space Γ_Λ is defined in the same way with replacing $\mathbb{R}^d$ onto Λ. The Lebesgue–Poisson measure λ on Γ_0 is defined as $\lambda := \sum_{n=0}^{\infty} \frac{1}{n!} m^{(n)}$. We preserve the same notation for the restriction of λ onto Γ_Λ.

The configuration space

$$\Gamma := \left\{ \gamma \subset \mathbb{R}^d \,|\, |\gamma_\Lambda| < \infty, \text{ for all } \Lambda \in \mathcal{B}_b(\mathbb{R}^d) \right\}, \tag{1}$$

is equipped with the vague topology. Here $\gamma_\Lambda := \gamma \cap \Lambda$ and $|\cdot|$ means cardinality of a finite set. Γ is a Polish space (see, e.g., [8]) and the corresponding Borel σ-algebra $\mathcal{B}(\Gamma)$ appears the smallest σ-algebra for which all mappings $\Gamma \ni \gamma \to |\gamma_\Lambda| \in \mathbb{N}_0$ are measurable.

Let $B_{\mathrm{bs}}(\Gamma_0)$ be the set of bounded measurable functions on Γ_0 with bounded support, i.e. $G \restriction_{\Gamma_0 \setminus B} = 0$ for some bounded $B \in \mathcal{B}(\Gamma_0)$. The latter means that there exists $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $N \in \mathbb{N}$ such that $B \subset \bigsqcup_{n=0}^{N} \Gamma_\Lambda^{(n)}$.

On Γ we consider the set of cylinder functions $\mathcal{F}L^0(\Gamma)$ as the set of all measurable functions F for which there exists $\Lambda = \Lambda_F \in \mathcal{B}_b(\mathbb{R}^d)$ such that $F(\gamma) = F \restriction_{\Gamma_\Lambda} (\gamma_\Lambda)$. We define the following mapping between functions on Γ_0, e.g. $B_{\mathrm{bs}}(\Gamma_0)$, and functions on Γ, e.g. $\mathcal{F}L^0(\Gamma)$:

$$KG(\gamma) := \sum_{\eta \Subset \gamma} G(\eta), \qquad \gamma \in \Gamma, \tag{2}$$

see e.g. [6, 14, 15]. The summation in the latter expression is taken over all finite subconfigurations of γ, which is denoted by the symbol $\eta \Subset \gamma$. The mapping K is linear, positivity preserving, and invertible, with

$$K^{-1} F(\eta) := \sum_{\xi \subset \eta} (-1)^{|\eta \setminus \xi|} F(\xi), \qquad \eta \in \Gamma_0. \tag{3}$$

Let $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ be the set of all probability measures μ on $(\Gamma, \mathcal{B}(\Gamma))$ which have finite local moments of all orders, i.e. $\int_\Gamma |\gamma_\Lambda|^n \mu(d\gamma) < +\infty$ for all $\Lambda \in \mathcal{B}_b(\mathbb{R}^d)$ and $n \in \mathbb{N}_0$. A measure ρ on $(\Gamma_0, \mathcal{B}(\Gamma_0))$ is called locally finite iff $\rho(B) < \infty$ for all bounded sets B from $\mathcal{B}(\Gamma_0)$. The set of such measures is denoted by $\mathcal{M}_{\mathrm{lf}}(\Gamma_0)$.

One can define a transform $K^* : \mathcal{M}^1_{\mathrm{fm}}(\Gamma) \to \mathcal{M}_{\mathrm{lf}}(\Gamma_0)$, which is dual to the K-transform, i.e., for every $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, $G \in \mathcal{B}_{\mathrm{bs}}(\Gamma_0)$ we have

$$\int_\Gamma KG(\gamma)\mu(d\gamma) = \int_{\Gamma_0} G(\eta)\,(K^*\mu)(d\eta).$$

The measure $\rho_\mu := K^*\mu$ is called the correlation measure of μ. By [6], for $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ and $G \in L^1(\Gamma_0, \rho_\mu)$ the series (2) is μ-a.s. absolutely convergent.

We define correlation functional $k_\mu : \Gamma_0 \to (0; +\infty)$ corresponding to the measure μ as the density (provided it exists): $k_\mu(\eta) := \frac{d\rho_\mu}{d\lambda}(\eta)$, $\eta \in \Gamma_0$. The correlation functional k_μ may be considered as the system of correlation functions $k_\mu^{(n)}$ corresponding to the restrictions $k_\mu \restriction_{\Gamma^{(n)}}$. These symmetric functions are well known in statistical physics, see e.g. [17, 18]. In applications a specially important role play correlation functions of the first and second orders: $k^{(1)}(x)$ and $k^{(2)}(x, y)$. These functions describe, respectively, the density of particles and pair correlations.

A measure $\mu \in \mathcal{M}^1_{\mathrm{fm}}(\Gamma)$ is called translation invariant if it is invariant with respect to shifts of configurations $\Gamma \ni \gamma \mapsto \{x + a \,|\, x \in \gamma\} \in \Gamma$ for any $a \in \mathbb{R}^d$. The first-order correlation function of such measure doesn't depend on the space coordinate: $k_\mu^{(1)}(x) \equiv k_\mu^{(1)}$; and the second-order correlation function depends on difference of coordinates: $k_\mu^{(2)}(x, y) = k_\mu^{(2)}(x - y)$.

3 Stochastic Development Models

Spatial birth-and-death processes describe dynamics of configurations in $\mathbb{R}^d$ when particles disappear (die) from configurations and, on the other hand, some new particles appear (born) somewhere in the space. The generator of spatial birth-and-death dynamics is heuristically given on measurable functions $F : \Gamma \to \mathbb{R}$ by

$$(LF)(\gamma) = \sum_{x\in\gamma} d(x, \gamma \setminus x)\big[F(\gamma \setminus x) - F(\gamma)\big]$$
$$+ \int_{\mathbb{R}^d} b(x, \gamma)\big[F(\gamma \cup x) - F(\gamma)\big]dx, \tag{4}$$

where $d, b : \mathbb{R}^d \times \Gamma \to [0, \infty]$ are measurable rates of death and birth respectively. Of course, these rates should be finite for a.a. $\gamma \in \Gamma$ with respect to a proper measure. Suppose that, additionally, $b(\cdot, \gamma) \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$ for a.a. $\gamma \in \Gamma$. Then this operator is well-defined at least on $\mathcal{F}L^0(\Gamma)$. Indeed, for $F \in \mathcal{F}L^0(\Gamma)$

$$F(\gamma \setminus x) - F(\gamma) = F(\gamma \cup x) - F(\gamma) = 0, \quad x \in \Lambda^c := \mathbb{R}^d \setminus \Lambda$$

(where Λ, depending on F, is from definition of $\mathcal{F}L^0(\Gamma)$), and the both terms of (4) are finite.

Note that if L is a generator of such process (even if we know that this process exists) then for the study of properties of the corresponding stochastic dynamics we need some information about the semigroup associated with L. This semigroup determines a solution to the Kolmogorov equation which has the following form:

$$\frac{dF_t}{dt} = LF_t, \quad F_t|_{t=0} = F_0. \tag{5}$$

In various applications the evolution of the corresponding correlation functions (or measures) helps already to understand the behavior of the process. The evolution of the correlation functions of the process is related to the evolution of states of the system. The latter evolution is given as a solution to the dual Kolmogorov equation:

$$\frac{d\mu_t}{dt} = L^*\mu_t, \quad \mu_t|_{t=0} = \mu_0, \tag{6}$$

where L^* is the adjoint operator to L on $\mathcal{M}^1_{\mathrm{fm}}(\Gamma)$, provided, of course, that it exists.

Using explicit form of $\hat{L} := K^{-1}LK$ we derive the evolution equation for *quasi-observables* (functions on Γ_0) corresponding to the Kolmogorov equation (5). It has the following form

$$\frac{dG_t}{dt} = \widehat{L}G_t, \quad G_t \big|_{t=0} = G_0. \tag{7}$$

Then in the way analogous to those in which (6) was determined for (5), we get an evolution equation for the correlation functions corresponding to (7):

$$\frac{dk_t}{dt} = \widehat{L}^*k_t, \quad k_t \big|_{t=0} = k_0. \tag{8}$$

The generator $\hat{L}^*$ here is the dual to $\hat{L}$ w.r.t. the duality given by the following expression:

$$\langle\langle G, k\rangle\rangle = \int_{\Gamma_0} G \cdot k \, d\lambda. \tag{9}$$

Free development model A simplest model in considered framework is a model of free development when particles are born independently without any influence of existing ones. An interpretation is that a "decision" about appearing of a new element is produced outside of the system and it is not motivated by the situation inside of the system. Moreover, in this simplest model particles (units) will not die.

The formal pre-generator of the Markov dynamics that describes such model is the following:

$$(L_\sigma F)(\gamma) = \sigma \int_{\mathbb{R}^d} [F(\gamma \cup x) - F(\gamma)]dx,$$

where $\sigma > 0$ is the intensity rate of new units creation.

The corresponding Markov process exists due to, e.g., [5]. Using results from [4], we obtain

$$\big(\hat{L}_\sigma G\big)(\eta) = \sigma \int_{\mathbb{R}^d} G(\eta \cup x)dx, \tag{10}$$

$$\big(\hat{L}^*_\sigma k\big)(\eta) = \sigma \sum_{x\in\eta} k(\eta \setminus x). \tag{11}$$

Immediately from (11) and (8) we conclude that the density of the free development model has the form

$$k_t^{(1)}(x) = k_0^{(1)}(x) + \sigma t.$$

Therefore, the density has linear growth in time. To prevent this growth we need to modify the generator introducing some regulation mechanisms in the model.

Development model with global regulation Below we consider a model with a global regulation reflected in the death rate by an assumption about a finite life time for units. More precisely, we assume that each point of the configuration has exponentially distributed (with some positive parameter m) random life time and these random times are independent. Hence, a death appears as a random event equally distributed for all units independently of their space locations.

A pre-generator describing such process has the following form:

$$(L_{\sigma,m}F)(\gamma) = m\sum_{x\in\gamma}\big[F(\gamma\setminus x) - F(\gamma)\big] + \sigma\int_{\mathbb{R}^d}\big[F(\gamma\cup x) - F(\gamma)\big]dx.$$

Note that the expression for $L_{\sigma,m}$ coincides with the one for the generator of so-called Surgailis process (see [12, 20, 21]). Again, using results from [4], we obtain

$$\big(\hat{L}_{\sigma,m}G\big)(\eta) = -m|\eta|G(\eta) + \sigma\int_{\mathbb{R}^d} G(\eta\cup x)dx, \tag{12}$$

$$\big(\hat{L}^*_{\sigma,m}k\big)(\eta) = -m|\eta|k(\eta) + \sigma\sum_{x\in\eta} k(\eta\setminus x). \tag{13}$$

The considered stochastic dynamics has a unique invariant measure which is the Poisson measure on Γ with constant intensity $\frac{\sigma}{m}$.

Using (13) one can obtain a precise expression for the density of the process:

$$k_t^{(1)}(y) = e^{-tm}k_0^{(1)}(y) + \frac{\sigma}{m}(1 - e^{-tm}).$$

Therefore, for an initial state with bounded density any positive global regulation rate m gives time-space bounded density which converges uniformly in space to the limiting Poisson density.

Establishment effects in the development model As we pointed out before, in the free development model an appearing of a new unit and its location are independent of the presented configuration of the system. A reasonable generalization of this model is such that the newborn unit prefers to choose a location with smaller density of already existing units. The latter may be considered as higher probability to survive in less occupied regions. The corresponding term which decreases the birth rate of the generator in densely populated areas is called the establishment term. We consider the special case when this rate has exponential form, but our considerations may be extended to more general establishment rates.

Let $0 \le \phi \in L^1(\mathbb{R}^d)$, $\phi(-x) = \phi(x)$, $x \in \mathbb{R}^d$, and

$$(L_\phi F)(\gamma) = \int_{\mathbb{R}^d} \exp\Big\{-\sum_{y\in\gamma}\phi(x - y)\Big\}[F(\gamma\cup x) - F(\gamma)]dx.$$

We suppose that there exist the dynamics of measures μ_t and let k_t will be corresponding correlation functions. Actually, the existence of a Markov process for considered case may be obtained from [5]. Using for any $\varphi \in C_0(\mathbb{R}^d)$ the equality

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^d} k_t^{(1)}(x)\varphi(x)\,dx = \frac{\partial}{\partial t}\int_{\Gamma}\langle\varphi,\gamma\rangle d\mu_t(\gamma) = \int_{\Gamma} L_\phi\langle\varphi,\gamma\rangle d\mu_t(\gamma)$$

we obtain, by Jensen's inequality,

$$\begin{aligned}\frac{\partial}{\partial t}k_t^{(1)}(x) &= \int_{\Gamma}\exp\left\{-\sum_{y\in\gamma}\phi(x-y)\right\}d\mu_t(\gamma)\\ &\geq \exp\left(-\int_{\Gamma}\sum_{y\in\gamma}\phi(x-y)d\mu_t(\gamma)\right)\\ &= \exp\left(-\int_{\mathbb{R}^d}\phi(x-y)k_t^{(1)}(y)dy\right).\end{aligned}$$

In the translation invariant case we obtain

$$\frac{d}{dt}k_t^{(1)} \geq \exp\bigl(-\langle\phi\rangle k_t^{(1)}\bigr),$$

where $\langle\phi\rangle = \int_{\mathbb{R}^d}\phi(x)\,dx$. Hence, if g_t is a positive solution of the equation

$$\frac{d}{dt}g_t = e^{-\langle\phi\rangle g_t},$$

then $k_0^{(1)} \geq g_0$ implies $k_t^{(1)} \geq g_t$. One has

$$\begin{aligned}g_t &= \frac{1}{\langle\phi\rangle}\ln(\langle\phi\rangle t + C), \quad C > 1;\\ g_0 &= \frac{1}{\langle\phi\rangle}\ln C.\end{aligned}$$

Putting for any $k_0^{(1)}$ the initial value $g_0 := k_0^{(1)}$ we obtain that

$$k_t^{(1)} \geq \frac{1}{\langle\phi\rangle}\ln\bigl(\langle\phi\rangle t + \exp\bigl\{k_0^{(1)}\langle\phi\rangle\bigr\}\bigr). \tag{14}$$

Therefore, the establishment term cannot prevent unboundedness of density. We may expect only essentially slower growth due to the establishment effect.

Remark 1 Of course, if we consider two regulation mechanisms, namely, the global regulation and the establishment together, then the first-order correlation function will be also bounded. Moreover, in this case the operator

$$\begin{aligned}(L_G F)(\gamma) &= m\sum_{x\in\gamma}\bigl[F(\gamma\setminus x) - F(\gamma)\bigr]\\ &\quad + \sigma\int_{\mathbb{R}^d}[F(\gamma\cup x) - F(\gamma)]\exp\left\{-\sum_{y\in\gamma}\phi(x-y)\right\}dx\end{aligned}$$

is the generator of so-called Glauber dynamics for a classical gas model (see, e.g., [10, 11]). If ϕ has some additional properties such that there exists Gibbs measure with this potential, then such measure will be invariant (and even symmetrizing one) for the generator L_G. On the other hand, known properties of the corresponding Markov dynamics imply that corresponding correlation functions satisfied so-called generalized Ruelle bounds (see [10] for details).

4 Stochastic Development Models with Competitions

In the previous section we considered, in particular, global (outward) regulation in the model. As wee see, such regulation may prevent unbounded (linear) growth (in time) of the density of our system. In this section we consider the case of a local regulation which appear due to the competition between elements (units) of the system. A pre-generator which describes such model has the following form:

$$(L_{a,\sigma}F)(\gamma) = \sum_{x\in\gamma}\Bigg(\sum_{y\in\gamma\setminus x} a(x-y)\Bigg)[F(\gamma\setminus x) - F(\gamma)] \\ + \sigma\int_{\mathbb{R}^d}[F(\gamma\cup x) - F(\gamma)]dx.$$

Here $0 \le a \in L^1(\mathbb{R}^d)$ is an even function such that

$$\langle a\rangle := \int_{\mathbb{R}^d} a(x)dx > 0.$$

The question about existence of a process with the generator $L_{a,\sigma}$ we will not discuss in this paper. We just assume that there exist the dynamics of measures μ_t and let k_t will be the corresponding correlation functional.

Using results from [4] we obtain that

$$\big(\hat{L}_{a,\sigma}G\big)(\eta) = -2E_a(\eta)G(\eta) - \sum_{x\in\eta}\Bigg(\sum_{y\in\eta\setminus x} a(x-y)\Bigg)G(\eta\setminus x) \\ + \sigma\int_{\mathbb{R}^d} G(\eta\cup x)dx$$

and

$$\big(\hat{L}^*_{a,\sigma}k\big)(\eta) = -2E_a(\eta)k(\eta) - \int_{\mathbb{R}^d}\sum_{y\in\eta} a(x-y)k(\eta\cup x)dx \\ + \sigma\sum_{x\in\gamma} k(\eta\setminus x), \qquad (15)$$

where we used the following notations for the energy functional corresponding to the pair potential $a(\cdot)$:

$$E_a(\eta) = \sum_{\{x,y\}\subset\eta} a(x-y), \qquad \eta\in\Gamma_0.$$

It is easy to see that the Cauchy problems (7) and (8) for quasi-observables and correlation functions respectively have a form of hierarchical chains and, therefore, can not be solved explicitly. The latter is a common problem in the study of stochastic dynamics of IPS. In several particular models such as Glauber type dynamics in continuum [9, 10] or some spatial ecological models [3] this difficulty may be overcame via a proper perturbation theory techniques. As a result, in the mentioned works we have existence results for corresponding evolutional equations together with certain a-priori bound for the solutions. Note that the perturbation techniques needs, in any case, a presence in the system a small parameter. In the considered model such parameter is clearly absent. Nevertheless, one can try to find estimate for the correlation functions. Actually, in the presented below approach we will use the explicit form of the Markov generator to obtain an a-priori bound on the density of the system.

We will say that a sequence $\{\Lambda_k,\ k\in\mathbb{N}\}$ of open bounded subsets of $\mathbb{R}^d$ is of *F-type* if $\bigcup_{k\in\mathbb{N}}\Lambda_k=\mathbb{R}^d$, $\Lambda_k\subset\Lambda_{k+1}$, $k\in\mathbb{N}$ and there exists $F>0$ such that for any $h\in(0;1)$ and for any $k\in\mathbb{N}$

$$s(\Lambda_k,h):=\frac{|\Lambda_k(h)\setminus\Lambda_k|}{|\Lambda_k|}\le F,$$

where

$$\Lambda_k(h):=\Big\{x:\inf_{y\in\Lambda_k}|x-y|<h\Big\}.$$

Here and below $|\Lambda|$ means Lebesgue measure of $\Lambda\in\mathcal{B}_b(\mathbb{R}^d)$.

A simple example of *F-type* sequence is the sequence of balls $\Lambda_k=B(0,k)$ with center at origin and radius $k\in\mathbb{N}$. Indeed, for any $h<1$

$$s(\Lambda_k,h)=\frac{|\Lambda_k(h)\setminus\Lambda_k|}{|\Lambda_k|}-1=\frac{(k+h)^d}{k^d}-1=\left(1+\frac{h}{k}\right)^d-1<2^d-1.$$

For any $\Lambda\in\mathcal{B}_b(\mathbb{R}^d)$ we will call *the average density* of the our system the following object

$$\rho_t^\Lambda:=\frac{1}{|\Lambda|}\int_\Lambda k_t^{(1)}(x)dx,$$

where $k_t^{(1)}$ is the first-order correlation functions (density) at moment $t\ge 0$.

Lemma 1 *Suppose that the function a is continuous and positive definite and the sequence $\{\Lambda_k,k\in\mathbb{N}\}$ is F-type. Then there exists $c>0$ such that for any open $\Lambda\in\{\Lambda_k,k\in\mathbb{N}\}$*

$$2E_a(\eta)\ge c\,\frac{|\eta|^2}{|\Lambda|},\qquad \eta\in\Gamma_\Lambda.$$

Proof In [16], it was shown that for any continuous positive definite function a the energy E_a is superstable, namely, for any open $\Lambda\subset\mathbb{R}^d$ and for any $\eta:=\{x_i\}_{i=1}^n\subset\Lambda$ the following inequality holds

$$2E_a(\eta)\ge\frac{n^2}{|\Lambda|}\frac{[\langle a\rangle-\delta(h)]^2}{[\langle a\rangle+\delta(h)+s(\Lambda,h)\langle a\rangle]},$$

where

$$\delta(h) = 2\int_{|x|>h} a(x)dx \geq 0.$$

Therefore, for any $\Lambda \in \{\Lambda_k, k \in \mathbb{N}\}$

$$2E_a(\eta) \geq \frac{n^2}{|\Lambda|}\frac{[\langle a\rangle - \delta(h)]^2}{[\delta(h) + (F+1)\langle a\rangle]} =: \frac{n^2}{|\Lambda|}c.$$

Let $h \in (0;\ 1)$ be such that

$$\langle a\rangle - \delta(h) = \int_{|x|\leq h} a(x)dx - \int_{|x|>h} a(x)dx \neq 0$$

(we may always choose such h since the first integral is an increasing function of h and the second one is a decreasing function). Stress that $c > 0$ and doesn't depend on Λ. □

Theorem 1 *Suppose that the function a is continuous and positive definite and the sequence* $\{\Lambda_k, k \in \mathbb{N}\}$ *is F-type*; *let c be as in Lemma 1. Suppose also that there exists* $D > \sqrt{\frac{\sigma}{c}}$ *such that* $\rho_0^{\Lambda_k} \leq D$, $k \in \mathbb{N}$. *Then for any* $t > 0$, $k \in \mathbb{N}$

$$\rho_t^{\Lambda_k} \leq D.$$

Proof Note that for $F(\gamma) = \langle \varphi, \gamma\rangle$, $\gamma \in \Gamma$, $\varphi \in C_0(\mathbb{R}^d)$ we have

$$(L_{a,\sigma}F)(\gamma) = -\sum_{x\in\gamma}\left(\sum_{y\in\gamma\setminus x} a(x-y)\right)\varphi(x) + \sigma\int_{\mathbb{R}^d}\varphi(x)dx.$$

Let $\varphi(x) = \mathbb{1}_\Lambda(x)$, $\Lambda \in \{\Lambda_k, k \in \mathbb{N}\}$. Then $F(\gamma) = |\gamma_\Lambda|$ and

$$\begin{aligned}(L_{a,\sigma}F)(\gamma) &= -\sum_{x\in\gamma}\left(\sum_{y\in\gamma\setminus x} a(x-y)\right)\mathbb{1}_\Lambda(x) + \sigma|\Lambda| \\ &= -\sum_{x\in\gamma_\Lambda}\left(\sum_{y\in\gamma\setminus x} a(x-y)\right) + \sigma|\Lambda| \\ &\leq -\sum_{x\in\gamma_\Lambda}\left(\sum_{y\in\gamma_\Lambda\setminus x} a(x-y)\right) + \sigma|\Lambda| \\ &= -2E_a(\gamma_\Lambda) + \sigma|\Lambda| \leq -\frac{c}{|\Lambda|}|\gamma_\Lambda|^2 + \sigma|\Lambda|.\end{aligned}$$

Let us set

$$\begin{aligned}n_t^\Lambda &:= \int_\Gamma |\gamma_\Lambda| d\mu_t(\gamma) = \int_\Gamma \langle \mathbb{1}_\Lambda, \gamma\rangle d\mu_t(\gamma) \\ &= \int_{\mathbb{R}^d} \mathbb{1}_\Lambda(x) k_t^{(1)}(x)dx = \int_\Lambda k_t^{(1)}(x)dx = |\Lambda|\rho_t^\Lambda.\end{aligned}$$

Then using Holder inequality

$$\begin{aligned}\frac{d}{dt}n_t^{\Lambda} &= \int_{\Gamma} L_{a,\sigma}|\gamma_{\Lambda}|d\mu_t(\gamma) \le \int_{\Gamma}\Big(\sigma|\Lambda| - \frac{c}{|\Lambda|}|\gamma_{\Lambda}|^2\Big)d\mu_t(\gamma)\\ &= \sigma|\Lambda| - \frac{c}{|\Lambda|}\int_{\Gamma}|\gamma_{\Lambda}|^2 d\mu_t(\gamma) \le \sigma|\Lambda| - \frac{c}{|\Lambda|}\Big(\int_{\Gamma}|\gamma_{\Lambda}|d\mu_t(\gamma)\Big)^2\\ &= \sigma|\Lambda| - \frac{c}{|\Lambda|}(n_t^{\Lambda})^2.\end{aligned}$$

As a result,

$$\frac{d}{dt}\rho_t^{\Lambda} \le \sigma - c\big(\rho_t^{\Lambda}\big)^2.$$

Therefore, if we consider the positive solutions of the Cauchy problem

$$\begin{cases}\frac{d}{dt}g(t) = \sigma - cg^2(t),\\ g(0) = g_0\end{cases} \tag{16}$$

with proper $g_0 > 0$ and if $\rho_0^{\Lambda} \le g_0$ then $\rho_t^{\Lambda} \le g(t)$, $t > 0$. Solving (16) we obtain

$$\ln\frac{|\sqrt{c}g(t) + \sqrt{\sigma}|}{|\sqrt{c}g(t) - \sqrt{\sigma}|} - \ln\tilde{C} = 2\sqrt{c\sigma}t, \quad \tilde{C} > 0;$$

$$g(t) = \frac{Ce^{2\sqrt{c\sigma}t}\sqrt{\sigma} + \sqrt{\sigma}}{Ce^{2\sqrt{c\sigma}t}\sqrt{c} - \sqrt{c}} = \sqrt{\frac{\sigma}{c}}\Big(1 + \frac{2}{Ce^{2\sqrt{c\sigma}t} - 1}\Big), \quad C \in \mathbb{R}.$$

Then

$$g(0) = \sqrt{\frac{\sigma}{c}}\Big(1 + \frac{2}{C-1}\Big), \quad C \in \mathbb{R}.$$

Let $g_0 = D \ge \rho_0^{\Lambda}$. Then since $D > \sqrt{\frac{\sigma}{c}}$ we have

$$C = \frac{2}{D\sqrt{\frac{c}{\sigma}} - 1} + 1 > 1$$

and

$$Ce^{2\sqrt{c\sigma}t} - 1 \ge C - 1 > 0.$$

As a result,

$$\rho_t^{\Lambda} \le g(t) \le \sqrt{\frac{\sigma}{c}}\Big(1 + \frac{2}{C-1}\Big) = D$$

for any $t > 0$ and for any $\Lambda \in \{\Lambda_k, k \in \mathbb{N}\}$. The statement is proved. □

Corollary 1 *Under conditions of Theorem 1 in the translation invariant case we have that* $k_0^{(1)} \le D$ *implies* $k_t^{(1)} \le D$.

At the end we consider a simple estimate for the second-order correlation function. Let us suppose that

$$a(u) > 0, \quad u \in \mathbb{R}^d.$$

Then in the translation invariant case the following estimate holds

$$\begin{aligned} k_t^{(2)}(u) &\le e^{-2a(u)t} k_0^{(2)}(u) + 2\sigma \int_0^t e^{-2a(u)(t-\tau)} k_\tau^{(1)} d\tau \\ &\le e^{-2a(u)t} k_0^{(2)}(u) + 2\sigma D \int_0^t e^{-2a(u)(t-\tau)} d\tau \\ &= e^{-2a(u)t} k_0^{(2)}(u) + \frac{\sigma D}{a(u)} \big(1 - e^{-2a(u)t}\big). \end{aligned}$$

We have two possible estimates

$$k_t^{(2)}(x-y) \le e^{-2a(x-y)t} k_0^{(2)}(x-y) + \frac{\sigma D}{a(x-y)} \tag{17}$$

and

$$k_t^{(2)}(x-y) \le e^{-2a(x-y)t} k_0^{(2)}(x-y) + C\sigma D t. \tag{18}$$

First of them may be useful for estimating of the second-order correlation function $k_t^{(2)}(x-y)$ on small distances between x and y uniformly by time. And the second one is such for estimating on big distances (however, non-uniformly by time).

Acknowledgements The financial support of DFG through the SFB 701 (Bielefeld University) and German-Ukrainian Project 436 UKR 113/94 is gratefully acknowledged. This work was partially supported by FCT, POCI2010, FEDER.

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